

A q -analogue of the Drinfeld-Sokolov hierarchy of type A and q -Painlevé system

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Abstract

In this article, we propose a q -analogue of the Drinfeld-Sokolov (DS) hierarchy of type A . We also discuss its relationship with the q -Painlevé VI equation and the q -hypergeometric function.

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1 Introduction

A relationship between Painlevé systems and infinite-dimensional integrable hierarchies has been studied. In a recent work [4, 13], we derive a coupled Painlevé VI system, which admits a hypergeometric solution [14, 16], from the DS hierarchy of type A . In this article, we give a q -difference analogue of the above result. Namely, we propose a q -analogue of the DS hierarchy of type A and derive a q -Painlevé system which admits a particular solution in terms of the q -hypergeometric function.

The DS hierarchies are extensions of the KdV hierarchy for affine Lie algebras [2]. For type $A_{N-1}^{(1)}$ among them, the hierarchies are characterized by partitions of a natural number N [5]. In this article, we propose a q -DS hierarchy corresponding to a partition (n^m) of mn . Note that its formulation is based on two preceding works [11], equivalent to the case $m = 1$, and [8], equivalent to the case $n = 1$.

The coupled Painlevé VI system given in [13] (or [15]) is derived from the DS hierarchy corresponding to a partition (n, n) by a similarity reduction.

Therefore, in this article, we investigate a similarity reduction of the q -DS hierarchy corresponding to a partition (n, n) ; we denote it by q - $P_{(n,n)}$. The system q - $P_{(2,2)}$ coincides with the q -Painlevé VI equation given by Jimbo and Sakai [7]. Thus we can regard q - $P_{(n,n)}$ as a higher order generalization of the q -Painlevé VI equation. The system q - $P_{(n,n)}$ also admits a particular solution in terms of the q -hypergeometric function ${}_n\phi_{n-1}$.

The q -Painlevé VI equation is a system of q -difference equations

$$\begin{aligned}\frac{x(t)x(q^{-1}t)}{\alpha_3\alpha_4} &= \frac{(y(q^{-1}t) - t\beta_1)(y(q^{-1}t) - t\beta_2)}{(y(q^{-1}t) - \beta_3)(y(q^{-1}t) - \beta_4)}, \\ \frac{y(t)y(q^{-1}t)}{\beta_3\beta_4} &= \frac{(x(t) - t\alpha_1)(x(t) - t\alpha_2)}{(x(t) - \alpha_3)(x(t) - \alpha_4)},\end{aligned}\tag{1.1}$$

where the parameters satisfy

$$\frac{\beta_1\beta_2}{\beta_3\beta_4} = q^{-1} \frac{\alpha_1\alpha_2}{\alpha_3\alpha_4}.$$

It is given as the compatibility condition of a system of linear q -difference equations

$$\begin{aligned}Y(q^{-1}z, t) &= \mathcal{A}(z, t)Y(z, t), \quad \mathcal{A}(z, t) = \mathcal{A}_0(t) + \mathcal{A}_1(t)z + \mathcal{A}_2(t)z^2, \\ Y(z, q^{-1}t) &= \frac{z(zI + \mathcal{B}_0(t))}{(z - q^{-1}t\alpha_1)(z - q^{-1}t\alpha_2)}Y(z, t).\end{aligned}\tag{1.2}$$

The coefficient matrices $\mathcal{A}_i(t)$ ($i = 0, 1, 2$) satisfy

$$\begin{aligned}\mathcal{A}_2(t) &= \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \quad \mathcal{A}_0(t) \text{ has eigenvalues } t\theta_1, t\theta_2, \\ \det \mathcal{A}(z, t) &= \kappa_1\kappa_2(z - t\alpha_1)(z - t\alpha_2)(z - \alpha_3)(z - \alpha_4),\end{aligned}$$

where

$$\kappa_1 = \frac{q}{\beta_3}, \quad \kappa_2 = \frac{1}{\beta_4}, \quad \theta_1 = \frac{\alpha_1\alpha_2}{\beta_1}, \quad \theta_2 = \frac{\alpha_1\alpha_2}{\beta_2}.$$

The dependent variables $x(t)$ and $y(t)$ are given by

$$x(t) = -\frac{(\mathcal{A}_0(t))_{12}}{(\mathcal{A}_1(t))_{12}}, \quad y(t) = \frac{q(x(t) - t\alpha_1)(x(t) - t\alpha_2)}{\mathcal{A}(x(t), t)}.$$

We derive the system (1.1) from q - $P_{(2,2)}$ with the aid of a q -Laplace transformation for a system of linear q -difference equations.

The q -hypergeometric function ${}_n\phi_{n-1}$ is defined by the power series

$${}_n\phi_{n-1} \left[\begin{matrix} \alpha_1, \dots, \alpha_{n-1}, \alpha_n \\ \beta_1, \dots, \beta_{n-1} \end{matrix} ; q, t \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1; q)_k \dots (\alpha_{n-1}; q)_k (\alpha_n; q)_k}{(\beta_1; q)_k \dots (\beta_{n-1}; q)_k (q; q)_k} t^k,$$

where $(\alpha; q)_k$ stands for the q -shifted factorial

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_k = (1 - \alpha)(1 - q\alpha) \dots (1 - q^{k-1}\alpha) \quad (k \geq 1).$$

Note that it reduces to the generalized hypergeometric function ${}_nF_{n-1}$ via a continuous limit $q \rightarrow 1$. We see that $x(t) = {}_n\phi_{n-1}$ satisfies a n -th order linear q -difference equation

$$\left((1 - q^{-1}\beta_1 T_{q,t}) \dots (1 - q^{-1}\beta_{n-1} T_{q,t})(1 - T_{q,t}) \right. \\ \left. - t(1 - \alpha_1 T_{q,t}) \dots (1 - \alpha_{n-1} T_{q,t})(1 - \alpha_n T_{q,t}) \right) x(t) = 0,$$

where $T_{q,t}$ stands for a q -shift operator such that $T_{q,t}x(t) = x(qt)$.

This article is organized as follows. In Section 2, we formulate a q -DS hierarchy and its similarity reduction corresponding to a partition (n^m) of mn . In Section 3, an explicit formula of q - $P_{(n,n)}$ is presented. We also discuss a group of symmetries and a Lax form for q - $P_{(n,n)}$. In Section 4, the q -Painlevé VI equation is derived from q - $P_{(2,2)}$. In Section 5, we show that q - $P_{(n,n)}$ admits a particular solution in terms of ${}_n\phi_{n-1}$.

Remark 1.1. *The DS hierarchy corresponding to a partition (n^m) have been studied in [3] from a viewpoint of a Hamiltonian structure. It seems to be reduced from the q -DS hierarchy given in this article via a continuous limit.*

Remark 1.2 ([17]). *The q -Painlevé VI equation (1.1) is also derived from q -UC hierarchy.*

2 q -DS hierarchy

In this section, we formulate a q -DS hierarchy and its similarity reduction corresponding to a partition (n^m) of mn .

Let λ and T_λ be a grading parameter and a corresponding q -shift operator such that $T_\lambda(\lambda) = q\lambda$. We define $mn \times mn$ matrices e_j, f_j, h_j ($j = 1, \dots, mn$) by

$$\begin{aligned} e_j &= E_{j,j+1} & f_j &= E_{j+1,j} & h_j &= E_{j,j} & (j \notin m\mathbb{Z}), \\ e_j &= \lambda E_{j,j+1} & f_j &= \lambda^{-1} E_{j+1,j} & h_j &= E_{j,j} & (j \in m\mathbb{Z}; j \neq mn), \\ e_{mn} &= \lambda E_{mn,1} & f_{mn} &= \lambda^{-1} E_{1,mn} & h_{mn} &= E_{mn,mn}, \end{aligned}$$

where $E_{i,j}$ stands for a matrix with 1 on the (i, j) -th entry and zeros elsewhere. We also set

$$e_{j,k} = e_j e_{j+1} \dots e_{j+k-1}, \quad f_{j,k} = f_{j+k-1} \dots f_{j+1} f_j,$$

where $e_{j+mn} = e_j$ and $f_{j+mn} = f_j$. Note that

$$T_\lambda(e_{j,m}) = qe_{j,m}, \quad T_\lambda(f_{j,m}) = q^{-1}f_{j,m} \quad (j = 1, \dots, mn).$$

Let us consider a set of grade one matrices which are mutually commutative. Explicitly, we take matrices

$$\Lambda_i = \sum_{j=1}^n e_{i+(j-1)m,m} \quad (i = 1, \dots, m).$$

Then we have

$$\Lambda_i \Lambda_j = \Lambda_j \Lambda_i = O, \quad T_\lambda(\Lambda_i) = q\Lambda_i \quad (i, j = 1, \dots, m; i \neq j).$$

Note that such choice of matrices is suggested by the previous work [1, 13, 9]. In the following, we formulate a q -DS hierarchy by using such matrices.

Let t_i and T_i ($i = 1, \dots, m$) be independent variables and corresponding q -shift operators such that

$$T_i(t_i) = qt_i, \quad T_i(t_j) = t_j \quad (j \neq i).$$

Under a condition $|q| > 1$, we consider a q -Sato equation

$$T_i(Z)Z^{-1} = T_i(W)(I - \varepsilon t_i \Lambda_i)W^{-1} \quad (i = 1, \dots, m), \quad (2.1)$$

with matrices of dependent variables

$$Z = \sum_{j=1}^{mn} z_{j,0} h_j + \sum_{j=1}^{mn} \sum_{k=1}^{\infty} z_{j,k} e_{j,k}, \quad W = I + \sum_{j=1}^{mn} \sum_{k=1}^{\infty} w_{j,k} f_{j,k},$$

where I stands for the identity matrix and $\varepsilon = 1 - q$. We assume that

$$z_{1,0} z_{2,0} \dots z_{mn,0} = 1. \quad (2.2)$$

Under the system (2.1), we set

$$\Psi = W \prod_{i=1}^m \prod_{k=0}^{\infty} (I - q^{-k-1} \varepsilon t_i \Lambda_i).$$

Then we obtain a system of linear q -difference equations called a Lax form

$$T_i(\Psi) = B_i \Psi \quad (i = 1, \dots, m), \quad (2.3)$$

where

$$B_i = T_i(W)(I - \varepsilon t_i \Lambda_i)W^{-1}.$$

The compatibility condition of the system (2.3) is equivalent to

$$T_i(B_j)B_i = T_j(B_i)B_j \quad (i, j = 1, \dots, m). \quad (2.4)$$

We call a system of q -difference equations (2.4) a q -DS hierarchy.

Next, we formulate a similarity reduction of the q -DS hierarchy. Under the system (2.4), we consider the following equation:

$$T_\lambda(W) = q^\rho T_{1,\dots,m}(W) q^{-\rho}, \quad \rho = \sum_{i=1}^m \sum_{j=1}^n \rho_i h_{i+(j-1)m}, \quad (2.5)$$

where $T_{i,\dots,m} = T_i T_{i+1} \dots T_m$ and $\rho_1 + \dots + \rho_m = 0$. Note that

$$\rho \Lambda_i = \Lambda_i \rho \quad (i = 1, \dots, m).$$

We set

$$\Psi = W \left(\prod_{i=1}^m \prod_{k=0}^{\infty} (I - q^{-k-1} \varepsilon t_i \Lambda_i) \right) \lambda^\rho.$$

Then we obtain a Lax form

$$T_\lambda(\Psi) = M\Psi, \quad T_i(\Psi) = B_i\Psi \quad (i = 1, \dots, m), \quad (2.6)$$

where

$$M = q^\rho T_{2,\dots,m}(B_1) T_{3,\dots,m}(B_2) \dots T_m(B_{m-1}) B_m.$$

Note that the matrix M is also given by

$$M = T_\lambda(W) q^\rho (1 - \varepsilon t_1 \Lambda_1) (1 - \varepsilon t_2 \Lambda_2) \dots (1 - \varepsilon t_m \Lambda_m) W^{-1}.$$

The compatibility condition of the system (2.6) is equivalent to

$$T_i(M)B_i = T_\lambda(B_i)M, \quad T_i(B_j)B_i = T_j(B_i)B_j \quad (i, j = 1, \dots, m). \quad (2.7)$$

We call a system of q -difference equations (2.7) a similarity reduction of the q -DS hierarchy.

Remark 2.1. The q -Sato equation (2.1) is derived as follows. Consider a matrix-valued function G defined by

$$G = G(t_1, \dots, t_m)[\lambda, \lambda^{-1}] = \left(\prod_{i=1}^m \prod_{k=0}^{\infty} (I - q^{-k-1} \varepsilon t_i \Lambda_i) \right) G(0, \dots, 0)[\lambda, \lambda^{-1}].$$

Then we obtain a system of q -difference equations

$$T_i(G) = (I - \varepsilon t_i \Lambda_i) G \quad (i = 1, \dots, m).$$

It implies the system (2.1) via a triangle decomposition $G = W^{-1}Z$. In addition, a time evolution of the q -DS hierarchy is given in a framework of the function G by

$$G = \left(\prod_{i=1}^m \prod_{j=1}^{l_i} \prod_{k=0}^{\infty} (I - q^{-k-1} \varepsilon t_{i,j_i} \Lambda_i) \right) G(0, \dots, 0)[\lambda, \lambda^{-1}].$$

Remark 2.2. Replacing $M \rightarrow I - \varepsilon M$ and $B_i \rightarrow I - \varepsilon t_i B_i$, we can rewrite the system (2.7) into

$$[D_\lambda - M, D_i - B_i] = 0, \quad [D_i - B_i, D_j - B_j] = 0 \quad (i, j = 1, \dots, m),$$

where

$$D_\lambda = \frac{1 - T_\lambda}{\varepsilon \lambda}, \quad D_i = \frac{1 - T_i}{\varepsilon t_i}.$$

Via a continuous limit $q \rightarrow 1$, it reduces to the similarity reduction of the DS hierarchy of type $A_{mn-1}^{(1)}$ given in [13].

3 Higher order q -Painlevé system

In this section, we present an explicit formula of the similarity reduction (2.7) corresponding to a partition (n, n) , where $n \geq 2$, in terms of dependent variables. We also discuss a group of symmetries and a Lax form for q - $P_{(n,n)}$.

Consider a Lax form

$$T_\lambda(\Psi) = M\Psi, \quad T_i(\Psi) = B_i\Psi \quad (i = 1, 2), \quad (3.1)$$

with $2n \times 2n$ matrices

$$\begin{aligned} M &= T_\lambda(W) q^\rho (1 - \varepsilon t_1 \Lambda_1) (1 - \varepsilon t_2 \Lambda_2) W^{-1}, \\ B_i &= T_i(W) (1 - \varepsilon t_i \Lambda_i) W^{-1} \quad (i = 1, 2), \end{aligned}$$

where

$$\Lambda_1 = \sum_{j=1}^n e_{2j-1,2}, \quad \Lambda_2 = \sum_{j=1}^n e_{2j,2}, \quad \rho = \sum_{j=1}^n \rho_1 (h_{2j-1} - h_{2j}),$$

and

$$W = I + \sum_{j=1}^{2n} \sum_{k=1}^{\infty} w_{j,k} f_{j,k}.$$

Then the compatibility condition of the system (3.1) is equivalent to the similarity reduction

$$T_1(B_2)B_1 = T_2(B_1)B_2, \quad T_i(M)B_i = T_\lambda(B_i)M \quad (i = 1, 2). \quad (3.2)$$

The matrices B_1 , B_2 and M can be expressed in terms of dependent variables $w_j = w_{j,1}$ and parameters κ_j ($j = 1, \dots, 2n$). We assume that indices of w_j are congruent modulo $2n$, namely $w_j = w_{j+2n}$. The equation (2.5) implies

$$T_{1,2}(w_{2j-1}) = q^{2\rho_1}w_{2j-1}, \quad T_{1,2}(w_{2j}) = q^{-2\rho_1-1}w_{2j} \quad (j = 1, \dots, n),$$

from which we obtain

$$\begin{aligned} M = & \sum_{j=1}^{2n} (1 - \varepsilon\kappa_j)h_j + \sum_{j=1}^n (q^{\rho_1}\varepsilon t_1 w_{2j} - q^{-\rho_1-1}\varepsilon t_2 w_{2j-2})e_{2j-1} \\ & + \sum_{j=1}^n (q^{-\rho_1}\varepsilon t_2 w_{2j+1} - q^{\rho_1}\varepsilon t_1 w_{2j-1})e_{2j} - q^{\rho_1}\varepsilon t_1 \Lambda_1 - q^{-\rho_1}\varepsilon t_2 \Lambda_2, \end{aligned}$$

and

$$B_i = \sum_{j=1}^{2n} u_{i,j}h_j + \sum_{j=1}^{2n} v_{i,j}e_j - \varepsilon t_i \Lambda_i \quad (i = 1, 2),$$

where

$$\begin{aligned} u_{1,2j-1} &= \frac{q^{-\rho_1}(1 - \varepsilon\kappa_{2j-1})}{1 + q^{-2\rho_1-1}\varepsilon t_2 w_{2j-2}T_1(w_{2j-1})}, \quad u_{1,2j} = 1 + \varepsilon t_1 T_1(w_{2j-1})w_{2j}, \\ v_{1,2j-1} &= \varepsilon t_1 w_{2j}, \quad v_{1,2j} = -\varepsilon t_1 T_1(w_{2j-1}), \\ u_{2,2j-1} &= 1 + \varepsilon t_2 T_2(w_{2j-2})w_{2j-1}, \quad u_{2,2j} = \frac{q^{\rho_1}(1 - \varepsilon\kappa_{2j})}{1 + q^{2\rho_1}\varepsilon t_1 w_{2j-1}T_2(w_{2j})}, \\ v_{2,2j-1} &= -\varepsilon t_2 T_2(w_{2j-2}), \quad v_{2,2j} = \varepsilon t_2 w_{2j+1}. \end{aligned}$$

Note that the equation (2.2) implies

$$\prod_{j=1}^{2n} (1 - \varepsilon\kappa_j) = 1, \quad \prod_{j=1}^{2n} u_{i,j} = 1 \quad (i = 1, 2).$$

We now assume that $t_2 = 1$. We also consider a change of variables

$$\begin{aligned} p &= q^n, \quad t = q^{2n\rho_1}t_1^n, \quad z = q^{-n(4\rho_1-n+1)/2}(\varepsilon\lambda)^n, \\ a_j &= q^{-\rho_1+j-1}(1 - \varepsilon\kappa_{2j-1}), \quad b_j = q^{-\rho_1+j-1}(1 - \varepsilon\kappa_{2j}), \\ x_j(t) &= q^{-2(j-1)\rho_1}t_1^{-j+1}w_{2j-1}, \quad y_j(t) = q^{(j-1)(2\rho_1+1)}t_1^j\varepsilon w_{2j}, \end{aligned}$$

for $j = 1, \dots, n$. Replacing $p \rightarrow q$, we can describe the similarity reduction (3.2) as follows.

Theorem 3.1. *The dependent variables $x_j(t)$, $y_j(t)$ ($j = 1, \dots, n$) satisfy a system of q -difference equations*

$$\begin{aligned} x_j(t) - x_{j-1}(t) &= \frac{a_j x_j(qt)}{1 + x_j(qt)y_{j-1}(t)} - \frac{b_{j-1}x_{j-1}(qt)}{1 + x_{j-1}(qt)y_{j-1}(t)}, \\ y_j(qt) - y_{j-1}(qt) &= \frac{b_j y_j(t)}{1 + x_j(qt)y_j(t)} - \frac{a_j y_{j-1}(t)}{1 + x_j(qt)y_{j-1}(t)}, \end{aligned} \quad (3.3)$$

for $j = 1, \dots, n$, where

$$b_0 = q^{-1}b_n, \quad x_0(t) = tx_n(t), \quad y_0(t) = q^{-1}t^{-1}y_n(t),$$

with relations

$$\prod_{j=1}^n a_j \frac{1 + x_j(qt)y_j(t)}{1 + x_j(qt)y_{j-1}(t)} = q^{(n-1)/2}.$$

We denote the q -Painlevé system (3.3) by q - $P_{(n,n)}$. As is seen in the next section, q - $P_{(2,2)}$ coincides with the q -Painlevé VI equation (1.1). Note that q - $P_{(n,n)}$ reduces to the coupled Painlevé VI system given in [13, 15] via a continuous limit $q \rightarrow 1$.

The system q - $P_{(n,n)}$ admits the affine Weyl group symmetry of type $A_{2n-1}^{(1)}$. Let r_j ($j = 0, \dots, 2n-1$) be birational transformations defined by

$$\begin{aligned} r_{2j-2}(a_j) &= b_{j-1}, \quad r_{2j-2}(b_{j-1}) = a_j, \\ r_{2j-2}(x_{j-1}(t)) &= x_{j-1}(t), \quad r_{2j-2}(y_{j-1}(y)) = y_{j-1}(t) + \frac{b_{j-1} - a_j}{x_j(t) - x_{j-1}(t)}, \\ r_{2j-2}(a_i) &= a_i, \quad r_{2j-2}(b_{i-1}) = b_{i-1}, \\ r_{2j-2}(x_{i-1}(t)) &= x_{i-1}(t), \quad r_{2j-2}(y_{i-1}(y)) = y_{i-1}(t) \quad (i \neq j), \end{aligned}$$

and

$$\begin{aligned} r_{2j-1}(a_j) &= b_j, \quad r_{2j-1}(b_j) = a_j, \\ r_{2j-1}(x_j(t)) &= x_j(t) + \frac{a_j - b_j}{y_j(t) - y_{j-1}(t)}, \quad r_{2j-1}(y_j(t)) = y_j(t), \\ r_{2j-1}(a_i) &= a_i, \quad r_{2j-1}(b_i) = b_i, \\ r_{2j-1}(x_i(t)) &= x_i(t), \quad r_{2j-1}(y_i(y)) = y_i(t) \quad (i \neq j). \end{aligned}$$

Then q - $P_{(n,n)}$ is invariant under actions of them. Furthermore the group of symmetries $\langle r_0, \dots, r_{2n-1} \rangle$ is isomorphic to the affine Weyl group of type $A_{2n-1}^{(1)}$.

Consider a gauge transformation for the Lax form (3.1)

$$\tilde{\Psi}(z, t) = \lambda^{-\rho_1} \sum_{j=1}^n q^{(j-1)(j-2)/2} ((\varepsilon t_1 \lambda)^{j-1} h_{2j-1} + (q^{-2\rho_1} \varepsilon \lambda)^{j-1} h_{2j}) \Psi.$$

Then we obtain a Lax form for q - $P_{(n,n)}$

$$\tilde{\Psi}(qz, t) = \tilde{M}(z, t) \tilde{\Psi}(z, t), \quad \tilde{\Psi}(z, qt) = \tilde{B}(z, t) \tilde{\Psi}(z, t), \quad (3.4)$$

with $2n \times 2n$ matrices

$$\begin{aligned} \tilde{M}(z, t) = & \sum_{j=1}^n a_j E_{2j-1, 2j-1} + \sum_{j=1}^n b_j E_{2j, 2j} + (y_1(t) - q^{-1} t^{-1} y_n(t)) E_{1, 2} \\ & + \sum_{j=1}^{n-1} (x_{j+1}(t) - x_j(t)) E_{2j, 2j+1} + \sum_{j=2}^n (y_j(t) - y_{j-1}(t)) E_{2j-1, 2j} \\ & + (x_1(t) - t x_n(t)) z E_{2n, 1} - \sum_{j=1}^{2n-2} E_{j, j+2} - t z E_{2n-1, 1} - z E_{2n, 2}, \end{aligned}$$

and

$$\begin{aligned} \tilde{B}(z, t) = & \sum_{j=1}^n \frac{a_j}{1 + x_j(qt) y_{j-1}(t)} E_{2j-1, 2j-1} + \sum_{j=1}^n (1 + x_j(qt) y_j(t)) E_{2j, 2j} \\ & + \sum_{j=1}^n y_j(t) E_{2j-1, 2j} - \sum_{j=1}^{n-1} x_j(qt) E_{2j, 2j+1} - t x_n(qt) z E_{2n, 1} \\ & - \sum_{j=1}^{n-1} E_{2j-1, 2j+1} - t z E_{2n-1, 1}, \end{aligned}$$

where $E_{i,j}$ stands for the matrix unit. The group of symmetries defined above is derived from gauge transformations for the Lax form (3.4)

$$r_j(\tilde{\Psi}(z, t)) = R_i(z, t) \tilde{\Psi}(z, t) \quad (j = 0, \dots, 2n-1),$$

where

$$\begin{aligned} R_0(z, t) &= I + \frac{b_n - q a_1}{x_1(t) - t x_n(t)} z^{-1} E_{1, 2n}, \\ R_{2j-2}(z, t) &= I + \frac{b_{j-1} - a_j}{x_j(t) - x_{j-1}(t)} E_{2j-1, 2j-2} \quad (j = 2, \dots, n), \\ R_{2j-1}(z, t) &= I + \frac{a_j - b_j}{y_j(t) - y_{j-1}(t)} E_{2j, 2j-1} \quad (j = 1, \dots, n). \end{aligned}$$

Note that a construction of those transformations is suggested by the previous works [10, 12].

4 q -Painlevé VI equation

The system q - $P_{(2,2)}$ is given as the compatibility condition of the Lax form

$$\Psi_4(qz, t) = M_4(z, t)\Psi_4(z, t), \quad \Psi_4(z, qt) = B_4(z, t)\Psi_4(z, t), \quad (4.1)$$

where

$$M_4(z, t) = \begin{bmatrix} a_1 & y_1(t) - \frac{y_2(t)}{qt} & -1 & 0 \\ 0 & b_1 & x_2(t) - x_1(t) & -1 \\ -tz & 0 & a_2 & y_2(t) - y_1(t) \\ \{x_1(t) - tx_2(t)\}z & -z & 0 & b_2 \end{bmatrix},$$

$$B_4(z, t) = \begin{bmatrix} \frac{qta_1}{qt+x_1(qt)y_2(t)} & y_1(t) & -1 & 0 \\ 0 & 1+x_1(qt)y_1(t) & -x_1(qt) & 0 \\ -tz & 0 & \frac{a_2}{1+x_2(qt)y_1(t)} & y_2(t) \\ -tx_2(qt)z & 0 & 0 & 1+x_2(qt)y_2(t) \end{bmatrix}.$$

In this section, we derive the system (1.2) from the Lax form (4.1) with the aid of a q -Laplace transformation (cf. [6, 8]).

First, we consider a gauge transformation

$$\Psi_4^*(z, t) = \tau_1(z, t)\Psi_4(z, t),$$

where $\tau_1(z, t)$ is a function such that

$$\frac{\tau_1(qz, t)}{\tau_1(z, t)} = \frac{1}{qa_1}, \quad \frac{\tau_1(z, qt)}{\tau_1(z, t)} = \frac{qt + x_1(qt)y_2(t)}{qta_1}.$$

Then the Lax form (4.1) is transformed into

$$\Psi_4^*(qz, t) = M_4^*(z, t)\Psi_4^*(z, t), \quad \Psi_4^*(z, qt) = B_4^*(z, t)\Psi_4^*(z, t), \quad (4.2)$$

where

$$M_4^*(z, t) = \frac{1}{qa_1}M_4(z, t), \quad B_4^*(z, t) = \frac{qt + x_1(qt)y_2(t)}{qta_1}B_4(z, t).$$

We set

$$M_4^*(z, t) = M_{4,0}^*(t) + zM_{4,1}^*(t), \quad B_4^*(z, t) = B_{4,0}^*(t) + zB_{4,1}^*(t).$$

Next, we consider a q -Laplace transformation

$$z\Psi_4^*(z) \rightarrow \frac{\Phi_4(\zeta) - \Phi_4(q^{-1}\zeta)}{\varepsilon\zeta}, \quad \Psi_4^*(qz) \rightarrow q^{-1}\Phi_4(q^{-1}\zeta),$$

where $\varepsilon = 1 - q$. Then the Lax form (4.2) is transformed into

$$\Phi_4(q^{-1}\zeta, t) = N_4(\zeta, t)\Phi_4(\zeta, t), \quad \Phi_4(\zeta, qt) = C_4(\zeta, t)\Phi_4(\zeta, t). \quad (4.3)$$

where

$$\begin{aligned} N_4(\zeta, t) &= (q^{-1}\varepsilon\zeta I + M_{4,1}^*(t))^{-1} (\varepsilon\zeta M_{4,0}^*(t) + M_{4,1}^*(t)), \\ C_4(\zeta, t) &= B_{4,0}^*(t) + \varepsilon^{-1}\zeta^{-1}B_{4,1}^*(t)(I - N_4(\zeta, t)). \end{aligned}$$

Denoting ζ^{-1} by z , we can restrict the Lax form (4.3) to the one with 3×3 matrices

$$\Psi_3(qz, t) = M_3(z, t)\Psi_3(z, t), \quad \Psi_3(z, qt) = B_3(z, t)\Psi_3(z, t), \quad (4.4)$$

thanks to the following lemma.

Lemma 4.1. *For each of the matrices $N_4(\zeta, t)$ and $C_4(\zeta, t)$, the first column is equivalent to the fundamental vector ${}^t[1, 0, 0, 0]$.*

Remark 4.2. *The Lax form (4.4) coincides with the one for the $q\text{-}\widehat{\mathfrak{gl}}_3$ hierarchy given in [8].*

In a similar way, we can reduce the Lax form (4.4) to the one with 2×2 matrices. We consider a gauge transformation

$$\Psi_3^*(z, t) = \tau_2(z, t)\Psi_3(z, t),$$

where $\tau_2(z, t)$ is a function such that

$$\frac{\tau_2(qz, t)}{\tau_2(z, t)} = \frac{a_1}{qb_1}, \quad \frac{\tau_2(z, qt)}{\tau_2(z, t)} = \frac{qta_1}{(qt + x_1(qt)y_2(t))(1 + x_1(qt)y_1(t))}.$$

We also consider a q -Laplace transformation

$$z\Psi_3^*(z) \rightarrow \frac{\Phi_3(\zeta) - \Phi_3(q^{-1}\zeta)}{\varepsilon\zeta}, \quad \Psi_3^*(qz) \rightarrow q^{-1}\Phi_3(q^{-1}\zeta).$$

Then the Lax form (4.4) is transformed into

$$\Phi_3(q^{-1}\zeta, t) = N_3(\zeta, t)\Phi_3(\zeta, t), \quad \Phi_3(\zeta, qt) = C_3(\zeta, t)\Phi_3(\zeta, t). \quad (4.5)$$

Denoting $\varepsilon^2 a_1 b_1 \zeta$ by z , we can restrict the Lax form (4.5) to the one with 2×2 matrices

$$\Psi_2(q^{-1}z, t) = M_2(z, t)\Psi_2(z, t), \quad \Psi_2(z, qt) = B_2(z, t)\Psi_2(z, t), \quad (4.6)$$

thanks to the following lemma.

Lemma 4.3. *For each of the matrices $N_3(\zeta, t)$ and $C_3(\zeta, t)$, the first column is equivalent to the fundamental vector ${}^t[1, 0, 0]$.*

The matrices $M_2(z, t)$ and $B_2(z, t)$ are of the form

$$M_2(z, t) = \frac{M_{2,0}(t) + zM_{2,1}(t) + z^2M_{2,2}(t)}{(z-t)(z-1)}, \quad B_2(z, t) = \frac{B_{2,0}(t) + zB_{2,1}(t)}{z-t},$$

where

$$M_{2,2}(t) = \frac{1}{b_1} \begin{bmatrix} a_2 & y_2(t) - y_1(t) \\ 0 & b_2 \end{bmatrix},$$

$$B_{2,1}(t) = \frac{1}{1 + x_1(qt)y_1(t)} \begin{bmatrix} \frac{a_2}{1+x_2(qt)y_1(t)} & y_2(t) \\ 0 & 1 + x_2(qt)y_2(t) \end{bmatrix}.$$

In order to derive the system (1.2), we consider a gauge transformation

$$Y(z, t) = \frac{(q^{-1}tz^{-1}; q^{-1})_{\infty}(qz; q)_{\infty}^2}{(q^{-1}z^{-1}; q^{-1})_{\infty}} \begin{bmatrix} \tau_3(t) & 0 \\ 0 & \tau_4(t) \end{bmatrix} \begin{bmatrix} 1 & \frac{y_2(t)-y_1(t)}{a_2-b_2} \\ 0 & 1 \end{bmatrix} \Psi_2(z, t).$$

where $\tau_3(z, t)$ and $\tau_4(z, t)$ are functions such that

$$\frac{\tau_3(qz, t)}{\tau_3(z, t)} = b_1, \quad \frac{\tau_3(z, qt)}{\tau_3(z, t)} = \frac{(1 + x_1(qt)y_1(t))(1 + x_2(qt)y_1(t))}{a_2},$$

$$\frac{\tau_4(qz, t)}{\tau_4(z, t)} = b_1, \quad \frac{\tau_4(z, qt)}{\tau_4(z, t)} = \frac{1 + x_1(qt)y_1(t)}{1 + x_2(qt)y_2(t)}.$$

Then the Lax form (4.6) is transformed into

$$Y(q^{-1}z, t) = \widetilde{M}_2(z, t)Y(z, t), \quad Y(z, t) = \frac{\widetilde{B}_2(z, q^{-1}t)}{z}Y(z, q^{-1}t). \quad (4.7)$$

Here the coefficient matrices satisfy

$$\widetilde{M}_2(z, t) = \widetilde{M}_{2,0}(t) + z\widetilde{M}_{2,1}(t) + z^2\widetilde{M}_{2,2},$$

$$\widetilde{M}_{2,2} = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}, \quad \widetilde{M}_{2,0}(t) \text{ has eigenvalues } ta_1, tb_1,$$

$$\det \widetilde{M}_2(z, t) = a_2b_2(z-t)(z-a_1b_1q^{-1/2}t)(z-1)(z-a_2^{-1}b_2^{-1}q^{1/2}),$$

and

$$\widetilde{B}_2(z, q^{-1}t) = \widetilde{B}_{2,0}(q^{-1}t) + zI_2,$$

$$\det \widetilde{B}_2(z, q^{-1}t) = (z - q^{-1}t)(z - q^{-1}a_1b_1q^{-1/2}t).$$

We can show them by direct computations; we do not state its detail here. By setting

$$\alpha_1 = 1, \quad \alpha_2 = \frac{a_1 b_1}{q^{1/2}}, \quad \alpha_3 = 1, \quad \alpha_4 = \frac{q^{1/2}}{a_2 b_2},$$

$$\beta_1 = \frac{b_1}{q^{1/2}}, \quad \beta_2 = \frac{a_1}{q^{1/2}}, \quad \beta_3 = \frac{q}{a_2}, \quad \beta_4 = \frac{1}{b_2},$$

we arrive at

Theorem 4.4. *The matrix-valued function $Y(z, t)$ solves the system (1.2).*

Corollary 4.5. *Under the system q - $P_{(2,2)}$, we set*

$$x(t) = \frac{t(x_2(t) - x_1(t))\xi_1(t)}{\xi_2(t)}, \quad y(t) = \frac{x_2(qt)(qt + x_1(qt)y_2(t))\psi_1(t)}{(1 + x_2(qt)y_2(t))\psi_2(t)},$$

where

$$\begin{aligned} \xi_1(t) &= qtx_1(t)y_1(t) - x_1(t)y_2(t) - qtx_2(t)y_1(t) + x_2(t)y_2(t) - (b_1 - a_1)qt, \\ \xi_2(t) &= (tx_2(t) - x_1(t))(x_2(t) - x_1(t))(y_2(t) - qty_1(t)) \\ &\quad + (b_1 - a_1)qtx_1(t) + \{(a_2 - b_1)t - (a_2 - a_1)\}qtx_2(t), \\ \psi_1(t) &= (1 - a_1 b_1 q^{1/2} t)x_2(qt)y_2(t) + qt - a_1 b_1 q^{1/2} t, \\ \psi_2(t) &= a_2(1 - a_1 b_1 q^{1/2} t)x_1(qt)x_2(qt)y_2(t) \\ &\quad + a_1(qt - a_2 b_1 q^{1/2} t)x_1(qt) + (a_2 - a_1)qtx_2(qt). \end{aligned}$$

Then those variables satisfy the q -Painlevé VI equation (1.1).

Note that q - $P_{(2,2)}$ gives explicit formulas of $x_1(qt)$ and $x_2(qt)$ by

$$x_1(qt) = \frac{-\xi_3(t)}{\xi_3(t)y_1(t) + (qt - a_1 b_1 q^{1/2} t)y_1(t) - (1 - a_1 b_1 q^{1/2} t)y_2(t)},$$

$$x_2(qt) = \frac{-\xi_4(t)}{\xi_4(t)y_1(t) + a_2(qt - a_1 b_1 q^{1/2} t)y_1(t) - a_2(1 - a_1 b_1 q^{1/2} t)y_2(t)},$$

where

$$\begin{aligned} \xi_3(t) &= a_1 q^{1/2} t(x_2(t) - x_1(t))(y_2(t) - y_1(t)) - qt + a_1 a_2 q^{1/2} t, \\ \xi_4(t) &= (x_2(t) - x_1(t))(y_2(t) - qty_1(t)) - b_1(qt - a_1 a_2 q^{1/2} t). \end{aligned}$$

5 q -Hypergeometric function ${}_n\phi_{n-1}$

In this section, we show that $qP_{(n,n)}$ admits a particular solution in terms of the q -hypergeometric function ${}_n\phi_{n-1}$.

Proposition 5.1. *Under the system $qP_{(n,n)}$, we consider a specialization*

$$y_j(t) = 0 \quad (j = 1, \dots, n), \quad \prod_{j=1}^n a_j = q^{(n-1)/2}.$$

Then a vector of the variables $\mathbf{x}(t) = {}^t[x_1(t), \dots, x_n(t)]$ satisfies a system of linear q -difference equations

$$\mathbf{x}(q^{-1}t) = \left(A_0 + \frac{A_1}{1 - q^{-1}t} \right) \mathbf{x}(t), \quad (5.1)$$

with $n \times n$ matrices

$$A_0 = \sum_{j=1}^n b_j E_{j,j} + \sum_{i=1}^n \sum_{j=i+1}^n (b_j - a_j) E_{i,j}, \quad A_1 = \sum_{i=1}^n \sum_{j=1}^n (a_j - b_j) E_{i,j}.$$

We always assume that

$$a_j \notin \mathbb{Z}, \quad a_i - a_j \notin \mathbb{Z} \quad (i, j = 1, \dots, n; i \neq j).$$

Note that A_0 is an upper triangular matrix and

$$A_0 + A_1 = \sum_{j=1}^n a_j E_{j,j} + \sum_{i=1}^n \sum_{j=1}^{i-1} (a_j - b_j) E_{i,j},$$

is a lower triangular matrix.

We consider a formal power series of $\mathbf{x}(t)$ at $t = 0$

$$\mathbf{x}(t) = t^{\log_q a_1} \sum_{k=0}^{\infty} (q^{-1}t)^k \mathbf{x}_k.$$

Recall that $|q| > 1$, namely $|q^{-1}| < 1$. Substituting it into the system (5.1), we obtain

$$(A_0 + A_1 - a_1 I) \mathbf{x}_0 = \mathbf{0}, \quad (A_0 + A_1 - a_1 q^{-k} I) \mathbf{x}_k = A_0 \mathbf{x}_{k-1} \quad (k \geq 1). \quad (5.2)$$

The matrices $A_0 + A_1 - a_1 I$ and $A_0 + A_1 - a_1 q^{-k} I$ are of rank $n - 1$ and n , respectively. Hence the recurrence formula (5.2) admits one parameter family of solutions. Its explicit formula is given by

$$\mathbf{x}_k = \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^k \begin{bmatrix} x_{k,1} \\ \vdots \\ x_{k,n} \end{bmatrix} \quad (k \geq 0),$$

where

$$x_{0,1} = 1, \quad x_{0,j} = \prod_{i=1}^{j-1} \frac{b_i - a_1}{a_{i+1} - a_1} \quad (j = 2, \dots, n),$$

and

$$x_{k,j} = \frac{(q^{\frac{a_1}{b_1}}; q)_k, \dots, (q^{\frac{a_1}{b_{j-1}}}; q)_k, (\frac{a_1}{b_j}; q)_k, \dots, (\frac{a_1}{b_{n-1}}; q)_k (\frac{a_1}{b_n}; q)_k}{(q^{\frac{a_1}{a_2}}; q)_k, \dots, (q^{\frac{a_1}{a_j}}; q)_k, (\frac{a_1}{a_{j+1}}; q)_k, \dots, (\frac{a_1}{a_n}; q)_k (q; q)_k} x_{0,j},$$

for $k \geq 1$. Then we arrive at

Theorem 5.2. *The system (5.1) admits a solution*

$$\mathbf{x}(t) = t^{\log_q a_1} \begin{bmatrix} c_1 \varphi_1(t) \\ \vdots \\ c_n \varphi_n(t) \end{bmatrix},$$

where

$$c_j = \prod_{i=1}^{j-1} \frac{b_i - a_1}{a_{i+1} - a_1},$$

$$\varphi_j(t) = {}_n\phi_{n-1} \left[\begin{matrix} q^{\frac{a_1}{b_1}}, \dots, q^{\frac{a_1}{b_{j-1}}}, \frac{a_1}{b_j}, \dots, q^{\frac{a_1}{b_{n-1}}}, \frac{a_1}{b_n} \\ q^{\frac{a_1}{a_2}}, \dots, q^{\frac{a_1}{a_j}}, \frac{a_1}{a_{j+1}}, \dots, \frac{a_1}{a_n} \end{matrix} ; q^{-1}, \frac{b_1 \dots b_n}{a_1 \dots a_n} q^{-1} t \right].$$

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